Simplified theoretical solution of circular toroidal shell with ribs under uniform external pressure

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Abstract

The toroidal shell with stiffened ribs is a new-style structure in ocean engineering especially in underwater engineering. This paper attempts to provide a simple theoretical method to obtain the stress solution of toroidal shell with ribs for its strength assessment. Firstly according to the structural property of toroidal shell with ribs and theory of curve-beam, a simple model for toroidal shell with ribs has been developed; then coupled with theory of thin-shell and elastic beam, its stress and deformation have been solved and can be expressed into analytic formulas; lastly by finite element method (FEM) and model experiment method, this simple theoretical solution has been verified to be reasonable and quite accurate. Thus this simple theoretical solution could be applied for analysis and design of pressure equipment in such toroidal structure type.

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1. Introduction

Compared to the cylindrical, spherical or conical shell, the toroidal shell is rarely used in underwater structures because of the difficulty of theoretical solution and the manufacture. However the type of toroidal shell is still widely used as joint components or accessories in pressure vessel and piping industry, nuclear power industry and ocean engineering for its specific structural shape to obtain kinds of performance. Especially Ross [1,2] recently suggested a new conceptual design of an underwater vehicle and an underwater space station with the main pressure hull in a toroidal shape, because of the advantages and disadvantages of using a circular cylindrical shell as pressure hull.
Many researchers in the field of engineering mechanics have spent great efforts to solve this problem in order to obtain the analysis and design method. Zhang [3], Clark [4] and Novozhilov [5] had obtained an elastic asymptotic solution of pure toroidal shells under axisymmetric loading. Moreover Xia [6] had obtained its elastic general solution due to arbitrary loading. Maching [7], Sobel and Flügge [8,9] had explored the buckling and stability of toroidal shells. Bushnel [10], Jordan [11] and Panagiotopoulos [12] had approached the stability analysis of the toroidal shells by finite difference method (FDM) or FEM. The buckling of the segments of toroidal shells had been solved by Stein and Mcelman [13] and Hutchinson [14]. Galletly [15,16] had studied the stability of closed toroidal shells with circular or non-circular cross-section. Blachut [17] had obtained three toroidal models experiments and revealed the collapse pressure by different material, geometry and manufacture, compared and confirmed with numerical results. Wang [18] had obtained the analysis of geometric nonlinear buckling and post-buckling of toroidal shells by asymptotic approach.

However all these previous works had been to solve the pure complete toroidal shell or its segments in elastic or nonlinear analysis, but then its critical pressure could not reach higher for non-ring-shaped ribs. Du [19] had studied static structural characteristics of a ring-stiffened toroidal shell with certain parameters by FEM and membrane simply theory, and then he [20] showed nonlinear structural properties of ring-stiffened toroidal shell by presented nonlinear FEM and verified the feasibility used as main pressure hull in underwater engineering. Zou and Du [21] obtained a theoretical method to calculate stress of ring-stiffened toroidal shell on two special positions that are on internal and external toroidal cycle. Du [22] also accomplished a steel welding ring-stiffened toroidal model experiment under hydrostatic external pressure, obtain its static elastic stress state and collapse pressure, compared with numerical results and revealed the buckling characteristics by nonlinear FEM.

In this paper the toroidal shell with ring-stiffened ribs will be solved in a theoretical method, which will be based on a simple curve-beam model with elastic supports on it. Then the stress and displacement of toroidal shell will also be obtained through by this solution of curve beam and thin shell theory. By numerical and model experimental method, this simple theoretical solution of toroidal shell with ring-stiffened ribs would be compared, certified and confirmed correctly finally.

2. Simple mechanical model for toroidal shell with ribs

The toroidal shell with stiffened ribs is difficult to solve by theoretical method, although all previous theoretical solution is just for pure toroidal shell without ribs. However when it has been set with series of stiffened ribs, the difficulty to solve it would become more and more than the former. Here it would be tried to accomplish such multiplex problem by a simplified theoretical method and shows out a simple curve-beam model.

2.1. Structural characteristic of toroidal shell under pressure

The toroidal shell is a special shells of revolution for the existence of horizontal tangents at φ = ±π/2, which are called the turning points. The Gaussian curvature ratio changes its sign from positive to negative (see from Fig. 1, where R represents the distance of the center of the meridian circles to the axis of rotation, φ is the tangential angle of shells, r is the curvature radius of the parallel circle on the toroidal shell to revolving direction, t is the wall thickness of shells, a is the radius of the meridian circle, θ is the rotational angle of the meridian circle).

The toroidal shell has nonlinear geography structure while the following characteristic of such type hull would expose this nonlinearity property. Here toroidal shell without ribs will be solved by FEM, including intensity and stability question. By all these ways, the structural characteristics of toroidal shell without ribs could have been discovered entirely.

The following toroidal model has dimensionless parameters as R/a=2.22, a/t=53, which are similar to those ever used in Ref. [20]. After geometric model being built and finite element meshed, the elastic statics analysis of toroidal shell under uniform external pressure load has been carried out. Based on the numerical calculation results, the stress strength coefficients distributing along section circle are shown out as Figs. 2 and 3. Here stress concentrated factor kp and ks will be defined as following by pressure and geometry sizes. Then membrane theory method for toroidal shell can be referred to our previous work [19].

\[
k_p = \frac{2\pi a}{pa} \quad k_s = \frac{2\pi a}{pa}
\]

From Fig. 2 and Fig. 3, it can be found that the stress factor kp and ks along section circle of toroidal shell is not linear or monotone but changing sharply at the turning point B.

Fig. 4 shows that the stress factor kp and ks vary by R/a, where models hold the same a/t. From this figure, it can be found that toroidal shell would be trend into cylindrical shell where R/a is infinity.

2.2. Establishment of simple mechanical model for toroidal shell with
ribs

According to structural property of toroidal shell with series of ribs and uniform external pressure load, it can be found that the structure shape and stress both vary along circle section between two stiffened ribs.

Similar to simplification of cylindrical shell with series stiffened ribs (Fig. 5), the curve beam model will be adopted to simplify toroidal shell with ribs to solve. It should be noticed that in Fig. 6, QQ' is an arbitrary arc along toroidal direction but between two neighbor ribs AC and A'C. This arc QQ' is curve beam in plane QO'O'.
3. Theoretical solution of toroidal shell with ribs

Based on the above curve-beam model, the strength of toroidal shell with stiffened ribs will be tried to be solved, and it would adopt some simplified model or some assumption in its derivation. Then finally the stress and displacement of toroidal shell can be expressed into detail formulas.

3.1. Governing equation for the problem

The toroidal shell is not axisymmetric after its ribs set up. However its mechanical model can be simplified into curve-beam model to solve and obtain stress and deformation.

The differential equation of curve beam is as follows:

\[ w + \frac{E \alpha^2}{2D} w + \frac{E \alpha^4}{2D^2} \frac{d^2 w}{d \phi^2} = \frac{F}{D} \left( 1 - \frac{n^2}{\alpha^2} \right) \frac{d^2 w}{d \phi^2} \]  

(2)

Define:

\[
\begin{align*}
2m &= \frac{E \alpha^2}{2D} \\
n^2 &= \frac{E \alpha^4}{2D^2} \\
k_1 &= \frac{E \alpha^4}{D} \left( 1 - \frac{n^2}{\alpha^2} \right) \\
r &= a + \frac{n}{\alpha \cos \phi}
\end{align*}
\]

Here \( E \) and \( \alpha \) is Young’s modulus and Poisson’s ratio, while \( D \) is bending stiffness of shell which is \( D = \frac{E \alpha^4}{12(1 - \nu^2)} \). Then

\[ w + 2mw + n^2 w = k_1 \frac{p}{D} + n^2 \frac{du}{d \phi} \]  

(3)

The boundary of curve beam with elastic support is as following:

\[
\begin{align*}
\theta &= \theta_0 \cos \phi, \quad w = 0 \\
ww &= \frac{d \phi}{k_2}, \\
\phi &= \frac{EF \alpha^4}{2D^2}
\end{align*}
\]

(4)

(5)

3.2. Solution of the governing equation

The homogeneous plus particular solution of Eq. (2) can be expressed as

\[ w = \frac{k_1}{n^2} + C_1 \sin \alpha_1 \cos \phi + C_2 \sin \alpha_2 \cos \phi + C_3 \sin \alpha_1 \sin \phi + C_4 \sin \alpha_2 \sin \phi \]  

(6)

Considering about the axisymmetric of curve beam at \( \theta = 0 \) and \( \partial \phi \partial \cos \phi \), which means the constant coefficient \( C_2 \) and \( C_3 \) must be zero. Thereby the above formula (6) can be simplified as following

\[ w = \frac{k_1}{n^2} + C_1 \sin \alpha_1 \cos \phi + C_4 \sin \alpha_2 \sin \phi \]  

(7)

Meanwhile it must be noticed that \( u \) is displacement of internal curve beam direction in toroidal shell, and based on its distribution along circle section plus that axisymmetric of toroidal shell. In derivation, we assume that \( u \) can be expressed by the function of \( \phi \), that is

\[ u = C_0 \frac{\phi}{\alpha} \sin \phi \sin \alpha_2 \phi \]  

(8)

According to the boundary (4) and (5) of curve beam and then especially in our derivation some minor items are ignored. So the coefficients \( C_1, C_4 \) and \( C_0 \) can be solved as following formulas.

\[
\begin{align*}
C_1 &= -\frac{2}{F} \left( \frac{k_1}{n^2} - \frac{d u}{d \phi} \right) (u_1 \sin \alpha_1 \cos \phi + u_2 \sin \alpha_2 \cos \phi) e_1 \\
C_4 &= -\frac{2}{F} \left( \frac{k_1}{n^2} - \frac{d u}{d \phi} \right) (u_1 \sin \alpha_1 \sin \phi - u_2 \sin \alpha_2 \sin \phi) e_1 \\
C_0 &= \frac{k_1}{F} (u_1 \sin \alpha_1 \cos \phi - u_2 \sin \alpha_2 \cos \phi) e_1
\end{align*}
\]

(9)

3.3. Stress and deformation of toroidal shell with ribs

Based on the above solution of curve-beam displacement, the stress of toroidal shell can be solved through by strain–displacement relation and stress–strain relation. So the stresses can be obtained by following formulas.

\[
\begin{align*}
s_\theta &= \frac{T_\theta}{I} + \frac{6 M_\theta}{l^2} \\
s_\phi &= E \nu_p + \mu s_\theta
\end{align*}
\]

(11)

Here

\[ s_\phi = \frac{1}{a} \frac{d w}{d \phi} - w \]

\[ M_\theta = D (s_\theta + \mu s_\phi) \]

\[ \kappa_\theta = \frac{w}{r^2}, \quad \kappa_\phi = \frac{1}{a^3} (w^* - u^*) \]
Therefore, the deformation and main stress at the end of curve-beam can be calculated by formula (12) and (13).

\[
\begin{align*}
\phi = \frac{d^2}{d\varphi^2}, \quad \psi = \frac{d}{d\varphi}, \quad \kappa = \frac{d}{d\varphi} \end{align*}
\]

By these two methods, the theoretical solution could be verified and confirmed presented theoretical method, while its nominal geometric parameters and material properties are shown as Table 1, where the local thickness \( t \) at special points such as point are provided at Table 2. The FEA model is respectively built according to its nominal parameters from Tables 1 and 2, and meanwhile its rigid displacement will be restricted.

The experimental model, some special points were set as measured points; put the strains, stuck and connected by wire lines, which will lead to machine for recording strain and pressure data during the process of hydrometrics pressure experiment, which details can be seen in Fig. 7 and those special points are annotated on model such as points A, C, M and M'. Here strain and stress measured in the second cycle step during hydrometrics pressure test will be used in the following section of this paper.

### 4. Verification of theoretical solution

To verify the curve beam model and this simplified theoretical solution, a steel welding model of toroidal shell with stiffened ribs is developed into being solved by numerical method and experiment too. Finite element method (FEM) is adopted to obtain the stress under some special pressure load, while experiment model is built and tested in external uniform pressure load from low to high pressure as model is still in elastic state. Therefore, the deformation and main stress at the middle of curve-beam can also be calculated by formula (14) and (15)

\[
\begin{align*}
\psi_{\text{nm}} &= - \frac{pa}{E} (1 + \varepsilon_1 - u) + \frac{pa^2}{Et} \\
\phi_{\text{nm}} &= - \mu \sigma_{\text{nm}}
\end{align*}
\]

Then from Eqs. (9) and (10), the normal displacement \( w \) and internal displacement \( u \) can be obtained too, so the stresses at the key position of curve beam may be solved totally from Eqs. (12)–(15).

### 4.1. Finite element analysis and experiment of toroidal model

Here a steel welding model of toroidal shell with stiffened ribs is experimental to verify and confirm presented theoretical method, while its nominal geometric parameters and material properties are shown as Table 1, where the local thickness \( t \) at special points such as point are provided at Table 2. The FEA model is respectively built according to its nominal parameters from Tables 1 and 2, and meanwhile its rigid displacement will be restricted.

In the experimental model, some special points were set as measured points, put the strains, stuck and connected by wire lines, which will lead to machine for recording strain and pressure data during the process of hydrometrics pressure experiment, which details can be seen in Fig. 7 and those special points are annotated on model such as points A, C, M and M'. Here strain and stress measured in the second cycle step during hydrometrics pressure test will be used in the following section of this paper. The test cycle steps are provided as Fig. 8.

### 4.2. Verification of the simple theoretical solution

Based on results from the elastic FEA and experiment of toroidal model with ribs, the stress of special key points can be compared with those results obtained by theoretical solution. Here it is noticed that the thicknesses at those different tested points are different, and so the measured thickness referred from Table 2 will be used when FE analysis and theoretical solution are executed at those special points. Table 3 and Table 4 have shown out stress concentration coefficients corresponding to those comparison results among these three different methods. Here the definitions of \( k_p \) and \( k_\varphi \) are same as previous formula (1), which means that the basic stress is membrane toroidal stress of the non-stiffened toroidal shell under external pressure \( p \).

From Table 3, it can be obviously found that this theoretical solution for the middle of two near ribs is both confirmed well by FEM and experiment results, and the maximum error is less than 5%.

---

### Table 1
Nominal parameters of toroidal model size and material.

<table>
<thead>
<tr>
<th>( R ) (mm)</th>
<th>( a ) (mm)</th>
<th>( t ) (mm)</th>
<th>( t_f = t_b ) (mm × mm)</th>
<th>( E ) (MPa)</th>
<th>( \nu )</th>
</tr>
</thead>
<tbody>
<tr>
<td>381</td>
<td>131.5</td>
<td>10</td>
<td>10 × 40</td>
<td>2.06 × 10^5</td>
<td>0.3</td>
</tr>
</tbody>
</table>
fened ribs and welding. The strains are dif-
may be caused by the quality of strain stuck on the shell, where
Comparison data of toroidal model at the end of ribs.
Table 4
Comparison data of toroidal model in the middle of two near ribs.
Table 3

<table>
<thead>
<tr>
<th>ϕ</th>
<th>kφ</th>
<th>kθ</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Outside</td>
<td>Inside</td>
</tr>
<tr>
<td>ϕ = 0 (Point M) Test (kθ)</td>
<td>–1.68</td>
<td>–1.77</td>
</tr>
<tr>
<td>Theory (kθ)</td>
<td>1.2</td>
<td>1.1</td>
</tr>
<tr>
<td>Δ%</td>
<td>2.4</td>
<td>1.1</td>
</tr>
<tr>
<td>ϕ = π (Point M) Test (kθ)</td>
<td>–1.38</td>
<td>–0.45</td>
</tr>
<tr>
<td>Theory (kθ)</td>
<td>1.41</td>
<td>–0.44</td>
</tr>
<tr>
<td>Δ%</td>
<td>2.2</td>
<td>2.2</td>
</tr>
<tr>
<td>FEM (kθ)</td>
<td>1.38</td>
<td>0.45</td>
</tr>
<tr>
<td>Δ%</td>
<td>1.4</td>
<td>0.5</td>
</tr>
</tbody>
</table>

From Table 4, it can also be found that theoretical solution for the end of two near ribs is confirmed by FEM and experiment results, and the maximum error is less than 10%. Here the error may be caused by the quality of strain stuck on the shell, where the strains are difficult to stick on ideal position because of stiffened ribs and welding.

Therefore, the curve beam model for toroidal shell with stiffened ribs is confirmed to be reasonable and theoretical solution of toroidal shell with ribs should be correct.

5. Conclusion

In this paper, the simple curve-beam model with elastic supports has been applied for the solution of toroidal shell with stiffened ribs under external pressure. By solving stress and displacement of curve-beam, the stress and displacement of toroidal shell with ribs have been obtained accordingly. Analysis results of simple theoretical solution have been confirmed by comparing with those of FEM and one steel welding model experiment. Therefore the theoretical solution has been confirmed correctly and could be adopted to design for underwater engineering, which could be convenient to provide stress factor curve for analysis and design.

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